Provision of a public good with altruistic overlapping generations and many tribes*

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Abstract

I develop a model to assess the relative importance, for the provision of a long-lived public good, of intergenerational altruism and contemporaneous cooperation. An application to climate policy suggests that contemporaneous cooperation is more important than altruism. Each of n tribes consists of a sequence of overlapping generations. Tribal members discount their own and their descendants' utility at different rates. Agents in the resulting game are indexed by their tribal affiliation and the time at which they act. The Markov Perfect equilibrium can be found by solving a control problem with a constant discount rate and an endogenous annuity.

Keywords: Overlapping generations, altruism, time consistency, Markov Perfection, differential games, climate policy

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1 Introduction

The non-cooperative provision of a long-lived public good depends on the ability of agents who make current decisions to cooperate with each other, and on the extent to which they care about the welfare of the not-yet born. The problem of climate change policy illustrates these two considerations, but similar issues arise in many other settings. I develop a model that helps to assess the relative importance, for the provision of a public good, of contemporaneous cooperation and intergenerational altruism. Using estimates of the costs and benefits of climate policy, I find that contemporaneous cooperation is likely to be much more important than intergenerational altruism in determining the equilibrium protection of the climate system.

Two decades of negotiation illustrate the difficulty of getting policymakers to cooperate on climate policy. A recent literature emphasizes the role of discounting in selecting climate policy (Nordhaus 2007), (Stern 2007), (Weitzman 2007). The UK and France currently use low social discount rates to evaluate long-lived public projects, and the US Environmental Protection Agency (2010) is considering a similar proposal. These reforms are intended to promote intergenerational equity. Even if such reforms were widely adopted, to what extent would they affect the equilibrium protection of the global commons, at given levels of international cooperation? How would increased international cooperation affect climate policy, for a given level of intergenerational altruism?

The climate problem is intergenerational. If agents currently alive discount their own future utility and the utility of the not-yet born at constant but different rates, then the aggregation of their preferences implies non-constant discounting, and time-inconsistent preferences. The existence of different tribes (countries, in the climate policy context) complicates the policy choice. I consider the case where there are $n \geq 1$ tribes, each of which has an equal share of the world's fixed population. The current population of each tribe consists of many generations. Members of a tribe care about their own utility stream and – to the extent that they have intergenerational altruism – about the utility streams of future tribal members, but they do not care about the utility streams of members of other tribes. Within each tribe and at each point in time, a social planner aggregates the preferences of tribal members currently alive. Because some agents die and new agents are born, the identity of this social planner changes over time.

When there is a single tribe in which agents discount their own and the unborn generations' utility at different rates, the result is the familiar problem of non-constant discounting (Strotz 1956), (Laibson 1997). When there are multiple tribes, each with a constant discount rate, the result is a differential game (Long 2010). With n tribes and non-constant discounting, the result is a game in which social planners are distinguished by the tribal index, i, and the time that she acts, t. The different tribes' social planners alive in the same period act simultaneously, and the social planner of any tribe acts before the future social planners. I consider a symmetric Markov Perfect Equilibrium (MPE), i.e. an equilibrium in which the optimal action of social planner i, t is a function of only the payoff-relevant state variable.

Most overlapping generations (OLG) models assume either that the individual lifetime is a random variable (Yaari 1965), (Blanchard 1985), (Calvo and Obstfeld 1988) or a finite known parameter, as in Diamond (1965)'s two period model or in Schneider, Traeger, and Winkler (2010)'s continuous time setting. I show that the discount rates in the two models, where agents' lifetimes are either exponentially distributed or a known constant, are always similar in the short run; the two rates might differ greatly at times close to or beyond the finite known lifetime. If current actions are insensitive to changes in distant discount rates, and if the deterministic lifetime is long, the two models are likely to be observationally similar. Thus, if the agents' lifetime corresponds to their biological span, the two models are likely to yield similar equilibria. If instead, the agents' lifetime corresponds to the time they are in political office, the two models may produce quite different equilibria.

A "paternalistically altruistic" agent cares about the utility flows enjoyed by future generations, but does not take into account that each of those agents also cares about the utility flows of subsequent agents. The "purely altruistic" agent does take into account the fact that her successors also value their own successors' utility flows (Ray 1987), (Andreoni 1989), (Saez-Marti and Weibull 2005). If lifetime is exponentially distributed, I find that the models of paternalistically and purely altruistic agents are observationally equivalent. The two types of agents have the same preferences if they use discount rates to evaluate their successors' payoffs that differ by the mortality rate.

After presenting the non-constant discount rates that arise in this OLG setting, I describe the game amongst the tribes and across the generations, and present the equilibrium conditions. Because the focus of this paper is to assess the relative importance of altruism and contemporaneous cooperation, in determining the equilibrium climate policy, I need to solve, rather than merely describe the equilibrium. For this reason, I use a linear-quadratic model, which involves only a few parameters and can be solved in closed form, modulo the solution to a cubic. This parsimony and transparency is particularly valuable in light of the uncertainty about the true costs and benefits of reducing carbon emissions. The linear-quadratic model has been widely used to study differential games (with a constant discount rate) in both industrial organization (Fershtman and Kamien 1987), (Reynolds 1987) and natural resource economics (Wirl 1994), (Dockner and van Long 1993); it has also been used to study quasi-hyperbolic discounting in the one-agent setting (Karp 2005).

2 Discounting

This section obtains the discount rates under different assumptions, for a representative tribe; thus, there is no tribal index here. Consider a public project, e.g. protection of the climate system. An agent's utility flow at a point in time depends on the current stock of the public good (e.g., the stock of greenhouse gasses) and on her tribe's current investment in that good (e.g., abatement). This investment cost is shared equally by all tribal members then alive, so *at time t they all have the same bounded flow of utility, u_t*. Agents' welfare consists of a selfish and an altruistic component. The selfish component equals the present discounted value of the expected flow of the agent's utility, using a constant pure rate of time preference. The altruistic component consists of the agent's evaluation of her successors' stream of utility. In this paper, "altruism" refers only to benevolence toward one's descendants, not toward current or future members of other tribes. At the cost of introducing another parameter, one could distinguish between intergenerational altruism within and across tribes.

The objective of this paper is to evaluate the relative influence, on investment in a public good, of agents' attitudes toward future generations and on the ability of different groups to cooperate at a point in time. In order to determine whether this comparison is sensitive to model details, I consider both paternalistic and pure altruism, and also the cases where agents' lifetime is exponentially distributed or a finite known number. In both of those cases, the population is fixed, so the birth and death rates are equal.

The memoryless feature of the exponential distribution means that all agents alive at a point in time have the same distribution function for their remaining lifetime. Because there is no private accumulation in this model, all agents alive at a point in time are identical. In this case, there is a representative agent in the usual sense.[1]

The paternalistic agent cares about her successors' utility steam, but does not take into account that each successor also cares about their own successors' utility stream. The purely altruistic agent does take into account the fact that all of her successors care about their own successors' utility streams. A description of preferences for the purely altruistic agent therefore requires a recursive model. For exponentially distributed lifetimes, the models with paternalistic and pure altruism are related by a simple change in the parameter that measures an agent's altruism.

I also consider a model of paternalistic altruism in which agents have a known, finite lifetime.[2] Here, agents alive at a point in time are different: the older ones will die sooner than the younger ones. In this setting, I assume that the representative agent (social planner) at a point in time is utilitarian; she puts equal weight on the preferences of all tribal members currently alive.

To the extent that an agent is altruistic, she cares about her tribal successors' utility streams. She either does not care, or acts as if she does not care, about currently living tribal members' utility streams. Bergstrom (2006) points out that when agents feel benevolence toward others who share both the costs and the benefits of a public good, it is necessary to count both the "sympathetic costs" as well as the "sympathetic benefits" (those arising from the feeling of benevolence). In the exponentially distributed case, where tribal members currently alive are identical, the sympathetic costs offset the sympathetic benefits, so benevolent feelings toward other tribal members currently living do not affect the cost benefit calculation. In the finitely lived case, agents are not identical, but other agents currently alive can speak for themselves; and the social planner gives each of their preferences equal weight. The unborn (future) tribal members enjoy the benefits of the public good but do not share the current investment costs, and they cannot directly influence the tribe's current social planner. Therefore, feelings of altruism toward these agents do affect the cost benefit calculation.

[1] Ekeland and Lazrak (2010) and Ekeland, Karp, and Sumaila (2012) contain some of the results reported below for the discount rate of the paternalistic exponentially-lived agent in continuous time. I repeat that material in order to make this paper self-contained.

[2] I have not yet analyzed the deterministic lifetime – pure altruism case.

2.1 The discrete time model

This section uses a discrete time model, in which each period lasts for ε units of time (e.g., years). It explains the assumptions and derives the discount factors for three cases: paternalistic and pure altruism with exponentially distributed lifetime, and paternalistic altruism with a fixed lifetime. Passing to the limit as $\varepsilon \to 0$ gives the continuous time formulae.

The pure rate of time preference that an agent uses to evaluate her selfish component of welfare is r. The "altruism parameter", λ, is the rate she uses to evaluate the utility of future generations. The corresponding one-period discount factors are $e^{-r\varepsilon}$ and $e^{-\lambda\varepsilon}$. For the case of exponentially distributed lifetime, θ is the mortality = birth = hazard rate, so $\gamma = r + \theta$ is the risk-adjusted discount rate and $e^{-\gamma\varepsilon}$ is the corresponding risk-adjusted discount factor. With a constant population normalized to 1, $1 - e^{-\theta\varepsilon}$ is the mass of agents who die, equal to the mass who are born, at the end of a period of length ε. In order to make the models with exponentially distributed and deterministic lifetimes comparable, I assume throughout that the lifetime in the latter case equals $T = \frac{1}{\theta}$, the expected lifetime under the exponential distribution. The rates r, θ, and λ are all positive, and restrictions discussed below imply bounded welfare.

2.1.1 Exponentially distributed lifetime, paternalistic altruism

The discount factor that agents born in any period use to evaluate their own future utility is $e^{-\gamma\varepsilon}$. All agents alive in a period are identical, so any can be chosen as the social planner who makes the decision about investment in the public good in that period. The social planner alive in period 0 gives weight $\left(1 - e^{-\theta\varepsilon}\right) e^{-\lambda\varepsilon\tau} e^{-\gamma\varepsilon(t-\tau)}$ to the period t utility of agents born in period $\tau \leq t$. There are $1 - e^{-\theta\varepsilon}$ of these agents, each of whom discounts her period-t utility at $e^{-\gamma\varepsilon(t-\tau)}$, and the current social planner values that utility at $e^{-\lambda\varepsilon\tau}$.[3] (An alternative discrete time model the produces the same continuous time limit sets the number of births and deaths at the end of a period equal to $\theta\varepsilon$ rather than $1 - e^{-\theta\varepsilon}$.)

The total weight that the current social planner puts on period-t utility flow (the discount factor) is the sum of the selfish and altruistic components

[3] An alternative discrete time model sets the number of births and deaths at the end of a period equal to $\theta\varepsilon$ rather than $1 - e^{-\theta\varepsilon}$. The limiting form of those two models, as $\varepsilon \to 0$ are the same, although they obviously differ for $\varepsilon > 0$.

for those currently alive:

$$D(t;\varepsilon) = e^{-\gamma\varepsilon t} + \left(1 - e^{-\theta\varepsilon}\right) \sum_{\tau=1}^{t} e^{-\lambda\varepsilon\tau} e^{-\gamma\varepsilon(t-\tau)} = \frac{\left(\left(e^{-\lambda\varepsilon} - e^{-\gamma\varepsilon}\right)e^{-\gamma\varepsilon t} + \left(1 - e^{-\theta\varepsilon}\right)e^{-\lambda\varepsilon}\left(e^{-\lambda\varepsilon t} - e^{-\gamma\varepsilon t}\right)\right)}{\left(e^{-\lambda\varepsilon} - e^{-\gamma\varepsilon}\right)}. \tag{1}$$

The first term on the right of the first line is the weight given to those currently alive and the second term is the weight given to those born between periods 1 and t. The second line assumes that $\gamma \neq \lambda$; the case $\lambda = \gamma$ follows from L'Hospital's Rule.

2.1.2 Exponentially distributed lifetime, pure altruism

The expected present value of the flow of utility of an agent alive in period τ (the selfish component of her welfare) is

$$\sum_{t=0}^{\infty} e^{-\gamma\varepsilon t} u_{\tau+t}.$$

An agent's total welfare equals the sum of her own selfish utility and the utility that she receives from the welfare of agents who are born in the future. Denote the total welfare of the agent born in period $\tau + t$ as $V_{\tau+t}$. In each period, $1 - e^{-\theta\varepsilon}$ agents are born. The agent currently alive attaches the weight $\left(1 - e^{-\theta\varepsilon}\right) e^{-\lambda\varepsilon t}$ to the welfare of the generation born t periods in the future. Thus, the welfare of the agent alive at period τ is[4]

$$V_\tau = \sum_{t=0}^{\infty} e^{-\gamma\varepsilon t} u_{\tau+t} + \left(1 - e^{-\theta\varepsilon}\right) \sum_{t=1}^{\infty} e^{-\lambda\varepsilon t} V_{\tau+t}. \tag{2}$$

The following proposition gives the formula for the discount factor D_s such that welfare V_τ equals the present discounted value of future utility flows u_s, i.e.:

$$V_\tau = \sum_{s=0}^{\infty} D_s u_{\tau+s} \quad \text{with} \quad D_0 = 1. \tag{3}$$

[4]Replacing the first term on the right side of equation (2) by $\tilde{u}_t \equiv \sum_{t=0}^{\infty} \alpha^t u_{\tau+t}$ and defining $a(t) = b\delta^t$, results in equation (2) of Saez-Marti and Weibull (2005).

Proposition 1 *Assume* $e^{-\lambda\varepsilon}\left(2-e^{-\theta\epsilon}\right)-e^{-\gamma\varepsilon}\neq 0$ *and* $\lambda>\theta$. *(i) The additively separable function in equation (3) equals the solution to the recursion in equation (2) if and only if the discount factor equals*

$$D(t;\varepsilon)=\frac{e^{-\gamma\varepsilon t}\left(e^{-\lambda\varepsilon}-e^{-\gamma\varepsilon}\right)+e^{-\lambda\varepsilon}\left(e^{-\lambda\varepsilon}\left(2-e^{-\theta\varepsilon}\right)\right)^{t}\theta\varepsilon}{e^{-\lambda\varepsilon}\left(2-e^{-\theta\varepsilon}\right)-e^{-\gamma\varepsilon}}. \tag{4}$$

(ii) (a) $D(t;\varepsilon)$ *is positive, bounded and approaches 0 as* $t\rightarrow\infty$*.(b)* $\sum_{s=0}^{\infty}D(s;\varepsilon)$ *is bounded, so* V *is bounded given that* u *is bounded.*

2.1.3 Deterministic lifetime, paternalistic altruism

Here, in contrast to the case of exponentially distributed lifetimes, agents alive in a period are different: the older ones will die sooner than the younger ones. I assume that the social planner in a period is utilitarian, i.e. she maximizes the sum of the discounted utility of those currently alive, plus the value that those agents give to the utility of the not-yet born.

Agents live for $T\geq 1$ periods.[5] Next period there will be $\left(1-\frac{1}{T}\right)$ of the original unit mass of agents still alive. The agents alive at $t=0$ discount their future utility using the selfish discount factor $e^{-r\varepsilon}$. The utilitarian social planner representing the agents alive at $t=0$ discounts their $t=1$ utility at $e^{-r\varepsilon}\left(1-\frac{1}{T}\right)$, which takes into account both the agents' impatience and the death of some of the agents alive at $t=0$. For $t\leq T-1$ the discount factor that this social planner uses to evaluate the future utility of agents currently alive is $e^{-r\varepsilon t}\left(1-\frac{t}{T}\right)$ and for $t\geq T$ the discount factor is 0, because all of the original agents will have died.

With a constant population, $\frac{1}{T}$ new agents are born in each period and the same number die. The agents alive in period 0 discount future generations' utility at $e^{-\lambda\varepsilon}$; for example, the weight that the agents alive in period 0 give to those born in year 1 is $\frac{e^{-\lambda\varepsilon}}{T}$, a factor that accounts for both the current agents' limited altruism and the fact that $\frac{1}{T}$ new agents arrive in each period. The agents who arrive in year 1 discount their own next period utility at $e^{-r\varepsilon}$, and because the agents alive in period 0 discount those agents' utility at $e^{-\lambda\varepsilon}$, the weight that the period 0 agents place on the $t=2$ utility of the agents who arrived at $t=1$ is $\frac{e^{-\lambda\varepsilon}e^{-r\varepsilon}}{T}$. Those alive at $t=0$ place

[5] Sumaila and Walters (2005) propose a similar model, but their formulae accounts for birth but not for death. With constant population, the discount factor must account for both death and birth.

the weight $\frac{e^{-\lambda\varepsilon i}e^{-r\varepsilon(t-i)}}{T}$ on the period t utility of agents who arrive in period i, for $t-T<i\leq t$.

The weight that the $t=0$ social planner places on the utility flow at period $t>0$ equals the sum of the weight that the agents whom she represents place on the selfish and altruistic components of their preferences. This sum has a different structure, depending on whether $t\leq T-1$ or $t\geq T$. For $\lambda\neq r$ the discount factor is

$$D(t;\varepsilon)=\begin{cases} e^{-r\varepsilon t}\left(1-\frac{t}{T}\right)+\frac{1}{T}\sum_{\tau=1}^{t}e^{-\lambda\varepsilon\tau}e^{-r\varepsilon(t-\tau)}= \\ \frac{1}{T}\left(\frac{e^{-\lambda\varepsilon}\left(e^{-\lambda\varepsilon t}-e^{-r\varepsilon t}\right)}{\left(e^{-\lambda\varepsilon}-e^{-r\varepsilon}\right)}+(T-t)\,e^{-r\varepsilon t}\right)\text{ for }t\leq T-1, \\ \frac{1}{T}\sum_{\tau=t-T+1}^{t}e^{-\lambda\varepsilon\tau}e^{-r\varepsilon(t-\tau)}= \\ \frac{1}{T}e^{-\lambda\varepsilon(t+1-T)}\frac{e^{-\lambda\varepsilon T}-e^{-r\varepsilon T}}{e^{-\lambda\varepsilon}-e^{-r\varepsilon}}\text{ for }t\geq T \end{cases} \tag{5}$$

For $T=2$, this model produces the familiar β,δ model of quasi-hyperbolic discounting (Laibson 1997). For $T=2$, the discount factor at $t=1$ is $\frac{e^{-r\varepsilon}+e^{-\lambda\varepsilon}}{2}$ and the discount factor at $t>1$ is $\frac{1}{2}e^{-\lambda\varepsilon(t-1)}\left(e^{-\lambda\varepsilon}+e^{-r\varepsilon}\right)$. Defining $\beta=\frac{e^{-r\varepsilon}+e^{-\lambda\varepsilon}}{2}$, $\delta=e^{-\lambda\varepsilon}$ produces the β,δ model.

2.2 Discounting in continuous time

This section presents and compares the continuous time discount factors and rates, obtained by taking the limit as $\varepsilon\rightarrow 0$ in the formulae above. Appendices A.2 and B.1 provide intermediate calculations and the proof of Proposition 2, and also acknowledge a slight abuse of notation: Section 2.1 uses t, τ and T as indices of the period number, whereas this section uses those variables to indicate time, measured in years.

2.2.1 Exponentially distributed lifetime

The continuous time discount factor for the paternalistic agent with exponentially distributed lifetime is

$$D(t)=\left(\frac{\lambda-r}{\lambda-\gamma}\right)e^{-\gamma t}-\frac{\theta}{\lambda-\gamma}e^{-\lambda t}. \tag{6}$$

The following proposition establishes an isomorphism between paternalistic and pure altruism.

Proposition 2 *Suppose that the agent with pure altruism discounts future agents' welfare at rate λ', while the agent with paternalistic altruism discounts future agents' utility at rate λ. Both have exponentially distributed lifetimes with mortality rate θ and the pure rate of time preference r. For the limiting case where $\varepsilon \to 0$, the two agents have the same preferences if and only if $\lambda' = \lambda + \theta$.*

Numerical examples establish that there is no analogous isomorphism in the discrete time setting, where $\varepsilon > 0$. The following corollary is a consequence of Proposition 2 and the fact that for $D(t)$ given by equation (6), $\frac{dD}{d\lambda} < 0$ for $t > 0$:

Corollary 1 *Given θ and the same preference parameters (r, λ), the agent with paternalistic altruism discounts the future flow of utility more heavily than the agent with pure altruism.*

This comparison is not surprising: the agent with pure altruism cares about future utility flows both because they affect the future generations that directly experience those flows, and because they affect the welfare of earlier generations that care about those future generations. In contrast, the agent with paternalistic altruism cares only about the direct affect of future utility flows on the agents who experience them.

In view of the isomorphism between the types of altruism, a separate treatment of the agent with pure altruism is unnecessary. The discount rate, η, associated with the discount factor in equation (6) is

$$\begin{aligned} -\tfrac{dD}{dt}\tfrac{1}{D} \equiv \eta(t) = \tfrac{-\gamma\lambda+\gamma r+\theta\lambda e^{-(\lambda-\gamma)t}}{-\lambda+r+\theta e^{-(\lambda-\gamma)t}} \Longrightarrow \\ \tfrac{d\eta}{dt} = -\theta e^{-t(\lambda-\gamma)}\left(r-\lambda\right)\tfrac{(\lambda-\gamma)^2}{\left(r-\lambda+\theta e^{t\gamma-t\lambda}\right)^2}. \end{aligned} \tag{7}$$

The definition of the discount rate $\eta(t)$ implies

$$\text{for } r < \lambda < \infty: \quad \frac{d\eta(t)}{dt} > 0; \quad \eta(0) = r; \quad \lim_{t\to\infty} \eta(t) = \gamma; \tag{8}$$

$$\eta_{|\lambda=\infty}(t) = \gamma \quad \eta_{|\lambda=r}(t) = r \quad \eta_{|\lambda=0}(t) = \gamma\frac{r}{r+\theta e^{\gamma t}}; \tag{9}$$

$$\text{for } 0 < \lambda < r: \quad \frac{d\eta(t)}{dt} < 0; \quad \eta(0) = r; \quad \lim_{t\to\infty} \eta(t) = \lambda. \tag{10}$$

Equation (8) states that if the current generation cares less about future generations' welfare than about its own, but has some concern for the future ($r < \lambda < \infty$), the discount rate increases over time from the pure rate of time preference, r, to the risk adjusted rate, $\gamma = r + \theta$. These limits are independent of λ. That is, for finite t, a finite value of $\lambda > r$ shifts down the entire trajectory of the discount rate, relative to the selfish rate. For $\lambda < r$, the discount rate falls, as with hyperbolic discounting (equation 10), and in this case the discount factor is a convex combination of two exponentials.

Equation (9) provides the discount rate for three limiting values of λ: ∞ (using the compactified real line), r, and 0. The fact that $\eta_{|\lambda=\infty}(t) = \gamma$ $\forall t \geq 0$ whereas $\eta(0) = r$ for $\lambda < \infty$ means that there is a discontinuity in $\eta(t)$ at $\lambda = \infty$.[6] The discount rate is constant over time for $\lambda = \infty$ and for the borderline case where the agent cares as much about future generations' utility as about her own ($\lambda = r$).

2.2.2 Finite lifetime, paternalistic altruism

The discount factor, $D(t)$, and discount rate, $\eta(t)$, in this model are:

$$D(t) = \begin{cases} e^{-rt}\left(\frac{1-e^{-(\lambda-r)t}}{T(\lambda-r)} + \frac{T-t}{T}\right) \text{ for } t \leq T \\ e^{-\lambda t}\left(e^{(\lambda-r)T} - 1\right)\frac{1}{T(\lambda-r)} \text{ for } t \geq T \end{cases} \tag{11}$$

$$\eta(t) = \begin{cases} \frac{\lambda e^{-(\lambda-r)t} - r(\lambda-r)(T-t) - \lambda}{e^{-(\lambda-r)t} - 1 - (\lambda-r)(T-t)} \text{ for } t \leq T \\ \lambda \text{ for } t \geq T \end{cases} . \tag{12}$$

The shape of the trajectory of the discount rate, for $t \in (0, T)$, depends on whether λ is greater or less than r. For $\lambda \in (r, \infty)$, the discount rate increases from r to λ as t increases. For $\lambda \in [0, r)$ the discount rate falls from r to λ, and for $\lambda = r$ the discount rate is constant. For $\lambda = 0$, the discount rate is $r^2 \frac{T-\tau}{e^{r\tau} + (T-\tau)r - 1}$ for $\tau \leq T$ and 0 for $\tau > T$. In the other limiting case, where $\lambda = \infty$, the discount rate is $\eta(t) = r + \frac{1}{T-t}$ for $0 \leq t < T$

[6] A calculation shows that for $\lambda > r$

$$\mu(\lambda) \equiv \int_0^\infty (\gamma - \eta(\tau))\, d\tau = \ln\left(\frac{\lambda - (r+\theta)}{\lambda - r}\right),$$

so $\lim_{\lambda\to\infty} \mu(\lambda) = 0$ and $\mu(\infty) = 0$: the $L1$ norm μ is continuous in λ.

and infinite for $T \geq 0$. As with exponentially distributed lifetimes, there is a discontinuity at $\lambda = \infty$.

2.2.3 Comparison and taxonomy

The following corollary summarizes the comparison of the discount rates under exponentially distributed and deterministic lifetimes for paternalistic agents.

Corollary 2 *(i) The discount rates in the two cases (exponentially distributed and deterministic lifetimes) are the same for all t iff $\lambda = r$ (where $\eta = r$). (ii) For $\lambda = \infty$ the discount rate under exponentially distributed lifetime is constant at $\gamma = r + \theta$, and the discount rate under a deterministic lifetime begins at $r + \frac{1}{T} = r + \theta$ and approaches ∞ as $t \to T$. (iii) For finite λ and for positive t close to 0 and for $t \geq T$, the discount rate for agents with exponentially distributed lifetime is greater than the discount rate for agents with deterministic lifetime if and only if $\lambda > r$. (iv) For $\lambda > r$ the difference between the discount rates approaches $\lambda - r - \theta$ as $t \to \infty$ and for $\lambda < r$ the difference approaches 0.*

Proof. Parts (i), (ii), (iv) and the claim in part (iii) regarding $t \geq T$ merely summarize the previous discussion. Verifying the claim in part (iii) regarding positive t close to 0 uses a third order expansion of the discount *factors* evaluated at $t = 0$: the discount factor with exponentially distributed lifetimes minus the discount factor with deterministic lifetimes equals $\frac{1}{6}\theta^2 (\lambda - r) t^3 + o(t^4)$. ■

This corollary suggests that in the cases where $\lambda \approx r$, or $\lambda < r$ or $\lambda \approx r + \theta$ the trajectories of the discount rates are "quite similar", so that the preferences of the two types of agents (with deterministic or exponentially distributed lifetimes) are also quite similar. However, if $\lambda >> r + \theta$ the discount rates, and thus preferences, are quite dissimilar in the two cases for $t \geq T$; for small t the discount rates are similar. If T is large, then the fact that the discount rates are dissimilar for $t \geq T$ is likely to be rather unimportant for current decisions. In summary, unless λ is large and T is small, the two models are "quite similar".

Figure 1 shows the discount rates in the two models, for $r = 0.02 = \theta$ (so $T = 50$). The increasing solid curve shows the discount rate under the exponentially distributed lifetime for $\lambda = 0.06$ and the decreasing solid

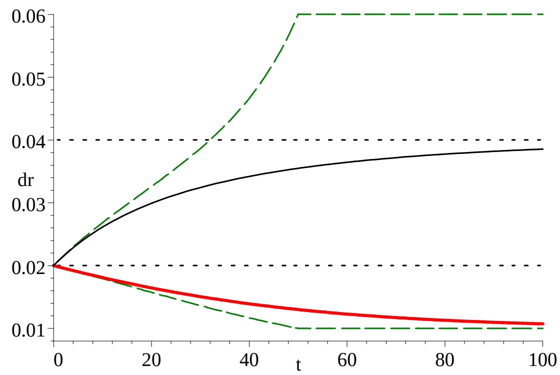

Figure 1: Exponentially distributed lifetime with $\theta = 0.02 = r$. Increasing solid curve shows discount rate for $\lambda = 0.06$ and decreasing solid curve shows discount rate for $\lambda = 0.01$. Finite lifetime $T = 50$ and $r = 0.02$. Increasing dashed curve shows discount rate for $\lambda = 0.06$ and decreasing dashed curve shows discount rate for $\lambda = 0.01$

curve shows the corresponding discount rate for $\lambda = 0.01$. The two dashed curves show the discount rates under the deterministic lifetime with $\lambda = 0.06$ (increasing) and $\lambda = 0.01$ (decreasing). This figure illustrates Corollary 2 and the comments that follow it.

The qualitative differences illustrated in Figure 1 follow naturally from the underlying assumptions of the two models. With exponentially distributed lifetime, the probability that an agent is alive at a point in time in the future decreases at a constant rate. Therefore, even if she cares nothing about the unborn generations, her discount rate for future utility flows never rises above $r+\theta$. In contrast, in the deterministic lifetime case, all agents currently alive will be dead after T years, so the utilitarian agent who aggregates the preferences of tribal members currently alive puts weight on utility flows after T years only to the extent that agents currently alive care about future generations.

Completely selfish agents put no weight on the welfare of the unborn ($\lambda = \infty$). A smaller value of λ implies a higher level of altruism. A plausible requirement of ethical behavior is to treat the present discounted value of the stream of utility of all agents symmetrically, regardless of when they are born (Ramsey 1928). That requirement implies $\lambda = 0$ (*not* $\lambda = r$)

which in turn implies that the discount rate converges to 0. I assume that λ is large enough that the equilibrium is well defined; that requirement is consistent with $\lambda = 0$ in the application is Section 4.

3 The game

Both the game and the equilibrium are easiest to describe in a discrete time setting, where the equilibrium conditions can be obtained using only elementary methods. The discrete time analog provides at least as good a representation of the world as does the continuous time version. For these reasons, I study the model by taking formal limits of a discrete time model, just as was done in Section 2.

There are n symmetric tribes, each of which consists of a sequence of overlapping generations of the type considered above. A larger value of n corresponds to greater fragmentation, or less cooperation, amongst agents at a point in time. At time t a social planner for tribe i chooses the action at time t, x_{it}. This action affects i's current utility flow and the evolution of a state variable, S_t, common to all tribes. For example, S_t is a climate-related variable, such as the stock of greenhouse gasses (GHG) or average temperature, and x_{it} is tribe i's GHG emissions or abatement at time t. In the interest of simplicity, I assume that tribe i's flow of utility at t, $u\left(S_t, x_{it}; n\right)$, depends only on the state variable and i's action. (Including j's current action in i's utility flow makes it possible to include leakage in a climate-related model or to consider the case of cross-tribal altruism.) In this symmetric environment, all tribes have the same utility function and discount factor. The state variable evolves according to

$$S_{t+\varepsilon} - S_t = f\left(S_t, \mathbf{x}_t; n\right)\varepsilon, \text{ with } \mathbf{x}_t = \left(x_{1t}, x_{2t} ... x_{nt}\right).$$

The parameter n in the growth function, f, and the utility function, u, makes it possible to consider a fragmentation of the economy (larger n) that leaves unchanged the set of feasible utility. Although a change in n has no intrinsic effect on aggregate (or per capita) utility and stock flows, a change in n does alter the equilibrium decisions, thereby altering the equilibrium aggregate utility and stock flows. That is, n has a strategic but not an intrinsic effect on outcomes.

The payoff to the social planner in tribe i at time t is the present discount value of the stream of utility:

$$\sum_{\tau=0}^{\infty} D(\tau;\varepsilon)u\left(S_{t+\tau}, x_{i,t+\tau}; n\right)\varepsilon, \tag{13}$$

where the discount factor depends on whether lifetime is exponentially distributed or deterministic, and whether altruism is pure or paternalistic. In this discrete time setting, agent it is the social planner in tribe i at period t. The strategic interactions in this game occur amongst different tribes and across periods. For all t, agents it and jt move simultaneously (choosing x_{it} and x_{jt}, respectively); for all i, j and for all $\tau > t$, agent it moves before agent $j\tau$.

I consider a symmetric stationary pure strategy Markov Perfect Equilibrium (hereafter, "MPE"), a function $\chi^{\varepsilon}(S)$ that satisfies the Nash condition: If the stock at t is S_t and agent it believes that agents jt, $\forall j \neq i$, use the decision rule $x_{jt} = \chi^{\varepsilon}(S_t)$, and that agents $j\tau$ $\forall j$ and $\tau > t$ will use the decision rule $x_{j\tau} = \chi^{\varepsilon}(S_\tau)$, then $x_{it} = \chi^{\varepsilon}(S_t)$ is the optimal action for agent it. The superscript ε applied to any function reminds the reader that this function corresponds to a particular value of ε, the length of a period in the discrete time setting.

The (assumed) symmetry of this problem makes it possible to consider a single tribe's problem, taking as given the behavior of all other tribes. Let the current period index be t, and suppose that $\forall \tau \geq t$ and $j \neq i$ agents $i\tau$ believe that agents $j\tau$ will use the policy $x_{j\tau} = \chi^{\varepsilon}(S_\tau)$. That is, each of the succession of social planners in tribe i expects each of the succession of social planners in other tribes to use this policy rule. With these beliefs, agents it expect the state to evolve according to

$$S_{\tau+\varepsilon} - S_\tau = F^{\varepsilon}(S_\tau, x_{i\tau})\varepsilon \text{ with } F(S_\tau, x_{i\tau}) \equiv f\left(S_\tau, \iota_{n-1}\chi^{\varepsilon}(S_\tau), x_{i\tau}; n\right), \tag{14}$$

$\forall\, \tau \geq t$, where ι_{n-1} is an $n-1$ dimensional vector consisting of 1's. If agents $i\tau$ $\forall \tau \geq t$ use the policy $\chi^{\varepsilon}(S_\tau)$, then the value of the program for the agent who faces the current state S is

$$J^{\varepsilon}(S) = \sum_{\tau=0}^{\infty} D(\tau;\varepsilon)u\left(S^*_{t+\tau}, \chi^{\varepsilon}\left(S^*_{t+\tau}\right); n\right)\varepsilon \tag{15}$$

where S^*_τ is the solution to equation (14) when all agents use the policy rule χ^{ε} and the initial condition is S.

The payoff in (13) with the equation of motion (14) are the ingredients of the game amongst the sequence of social planners in tribe i. These planners play a game rather than solving a standard control problem, because the function $D(\tau;\varepsilon)$ is not exponential: as a consequence, the sequence of actions that is optimal for an agent at a point in time is not optimal for her successors. The social planner at a point in time can choose the action at that time, but cannot commit her successors to particular actions. She therefore chooses the action that maximizes the discounted flow of current and future utility, understanding that future actions are chosen according to χ^{ε}.

The continuous time payoff and equation of motion for tribe i are

$$\int_{\tau}^{\infty} D(t-\tau)u(S_t, x_{it})dt \text{ and } \frac{dS_t}{dt} \equiv \dot{S}_t = F(S, x_i)$$
$$\text{with } F(S,x) \equiv f\left(S, \iota_{n-1}\chi\left(S\right), x_i; n\right),$$

where $\chi\left(S\right)$ is a MPE policy function in the continuous time setting, and the discount factor, D, is given by equation (6) in the case of agents with exponentially distributed lifetime, and by equation (11) in the case of agents with deterministic lifetime.

A larger value of n means that the time t social planner in a particular tribe internalizes a smaller fraction of the effect of her actions. A larger value of λ means that this social planner internalizes a smaller fraction of the long run effect of her actions. The point of this paper is to determine the relative sensitivity of equilibrium investments in a public good, to changes in n and λ.

3.1 Equilibrium conditions

Karp (2007) finds the formal limit, as $\varepsilon \to 0$, of the equilibrium conditions to the sequential game defined by the payoff in (15) with the equation of motion (14) for the one-tribe ($n = 1$) case. These conditions require that $J\left(S\right) = \lim_{\varepsilon\to 0} J^{\varepsilon}\left(S\right)$ and its first derivative exist – an assumption that can be checked given a particular equilibrium. These conditions can be applied directly to the case of $n > 1$: when the succession of planners in tribe i take as given other tribes' policy rule, tribe i has a standard problem with non-constant discounting. The resulting simplicity relies on the assumption of a symmetric (across both tribe and time indices) equilibrium. The models with $n = 1$ and $n > 1$ nevertheless differ; in the latter, the function $F(S, x) =$

$f\left(S, \iota_{n-1}\chi\left(S\right), x; n\right)$ depends on the policies of the other $n-1$ agents; those agents do not exist if $n = 1$.

The equilibrium conditions differ slightly in the two cases corresponding to $\lambda < r$ and $\lambda > r$ with exponentially distributed lifetime and in the case where agents have deterministic lifetimes. For $\lambda = r$ with both deterministic and exponentially distributed lifetimes, and for $\lambda = \infty$ with exponentially distributed lifetime, the discount rate is constant. In these cases, the tribes play a standard differential game, i.e. one without the strategic interactions within a tribe, across periods. I provide details for the model with exponentially distributed lifetime and $r < \lambda < \infty$, relegating the other two cases to Appendix B.2.

Dropping the agent index i (because of symmetry) and the superscript ε (because I now consider the continuous time limit) Proposition 1 and Remark 1 of Karp (2007) imply that $\chi\left(S\right)$ satisfies the necessary condition to the following fictitious optimal control problem with constant discount rate $\gamma = r + \theta$.

$$J(S) = \max \int_0^\infty e^{-\gamma t}\left(u(S_t, x_t) - K(S_t)\right) d\tau \quad \text{subject to } \dot{S} = F(S, x), \quad (16)$$

with the side condition (definition):

$$K(S_t) \equiv \int_0^\infty D(\tau)\left(\eta(\tau) - \gamma\right) u(S_\tau^*, \chi\left(S_\tau^*\right)) d\tau. \quad (17)$$

The tribe's utility flow on the equilibrium path is $u(S_\tau, \chi\left(S_\tau\right)$, the discount rate $\eta\left(\tau\right)$ is given by equation (7), and S_τ^* is the solution to the differential equation in (16) when all agents use the decision rule $\chi\left(S\right)$.

Given differentiability of $J(S)$ (already assumed in deriving the problem comprised of (16) and (17)), a necessary condition for the MPE is that

$$x_t = \chi\left(S_t\right) \equiv \arg\max\left(u(S_t, x_t) - K(S_t) + J_S\left(S\right) F(S, x)\right), \quad (18)$$

and that $J(S)$ satisfy the dynamic programming equation

$$\gamma J(S) = \left(u(S, \chi\left(S\right)) - K(S) + J_S\left(S\right) F(S, \chi\left(S_t\right))\right). \quad (19)$$

Appendix B.3 explains why these necessary conditions, together with the definition in equation (17), are also sufficient.

Equation (6) and the first line of equation (7) imply

$$D(t)(\eta(t)-\gamma) = -\theta e^{-\lambda t},$$

so equation (17) simplifies to

$$K(S_t) = -\theta \int_0^\infty e^{-\lambda\tau} u\left(S^*_{t+\tau}, \chi\left(S^*_{t+\tau}\right)\right) d\tau. \tag{20}$$

The integral in equation (20) is the present discounted value of the equilibrium future flow of payoff, computed using the discount rate λ. Thus, $-K(S_t)$ is an annuity, which if received in perpetuity and discounted at θ (the constant birth = death rate), equals the value of this future stream of payoff. The flow payoff in the fictitious control problem equals the flow playoff in the original model, plus this annuity.

3.2 Nonuniqueness

In general, the equilibrium to this model is not unique. Tsutsui and Mino (1990) note the existence of a continuum of stable steady states in the differential game with constant discounting; for each of these steady states there is an equilibrium policy function, defined at least in the neighborhood of the steady state. The economic explanation for this multiplicity (in the differential game) is that the decision whether to remain in a particular steady state depends on an agent's beliefs regarding the actions that rivals would take if a single agent were to drive the state away from that steady state. The (Markov perfect) equilibrium conditions do not pin down these beliefs. In a standard optimal control problem, the envelope theorem eliminates that kind of consideration, because the first order welfare effect of a deviation from the steady state is 0. This theorem is not applicable in the differential game. In this sense, the transversality condition typically applied to study autonomous control problem is "incomplete".

The same kind of consideration applies in the one-tribe model with non-constant discounting. Karp (2007) and Ekeland and Lazrak (2010) characterize the set of stable steady states in this setting. When $n > 1$ and the discount rate is non-constant, there are thus two sources of multiplicity of steady states arises, so we would expect the equilibrium to be unique (within the class that induce differentiable value functions) in only special circumstances.

Most applications use specific functional forms, because of the difficulty of obtaining general results outside the steady state. Models with isoelastic utility and linear or isoelastic growth functions have been used to study both differential games (Levhari and Mirman 1980) and the non-constant discounting problem with $n = 1$ (Barro 1999). The Introduction notes the widespread use of the linear-quadratic model. These models have a linear equilibrium. Several of these papers, and some others (Vieille and Weibull 2009), discuss the multiplicity of equilibria. This multiplicity is a consequence of the infinite horizon setting. In many cases, the linear policy is the unique equilibrium for the finite horizon version of these models, thus justifying the emphasis on this equilibrium.

Ekeland, Karp, and Sumaila (2012) study the $n = 1$ case using a model that is linear in the control variable, but otherwise allows general functional forms. That paper uses the exponentially distributed lifetime model, but all of its results carry over to the model with deterministic lifetime. Within the class of equilibria that gives rise to a differentiable value function, the equilibrium is unique: the stock follows a most rapid approach path to a steady state, which for $\lambda < \infty$ is independent of λ.

This linear-in-control model also has a simple equilibrium for $n > 1$, which consists of driving the state variable (e.g. the stock of fish) to a level at which the flow of utility is 0. If all other agents use a bang-bang control to drive the state to particular steady state S_∞ at which the flow of utility is strictly positive, then agent i has an incentive to undercut them (e.g. to harvest more than the level that sustains that stock). Thus, the only steady state is a level associated with a zero utility flow. In this example, the equilibrium is completely insensitive to (finite) λ and extremely sensitive to a change from $n = 1$ to $n = 2$.[7]

3.3 Interpretation of this game

Game theoretic models are better suited to asking what does happen, rather than what should happen. However, most game theoretic models – and

[7] In a MPE to the linear-in-control model, an agent knows that if she were to drive the state away from a locally stable steady state, the equilibrium response of other agents is to drive it back to this steady state *as rapidly as possible*. Consequently, the indeterminacy in beliefs that gives rise to the multiplicity of stable steady states in the general setting, is absent here. These remarks rest on the requirement that the value function is differentiable; with more general policies, many other MPE can exist (Dutta and Sundaram 1993).

certainly this one – represent such a high degree of abstraction that their predictive power, and thus their positive application, is limited. Game theoretic models help us to understand agents' incentives in circumstances that are too complicated to rely on intuition. Here the goal is to understand how limited cooperation amongst contemporaneous agents, and different levels of intergenerational altruism, interact to affect equilibrium decisions.

For $n = 1$, there is no conflict amongst contemporaneous agents and for $\lambda = r$, the time inconsistency problem also vanishes. Although $n = 1$, $\lambda = r$ leads to a standard optimization problem, rather than a game, this case does not provide a reasonable normative model because of the lack of an ethical basis for choosing $\lambda = r$. In evaluating a current investment decision, why should the utility stream of an agent born ten years from now matter more than the stream of an agent born twenty years from now?

The choice $\lambda = 0$ has a stronger ethical basis, and therefore (together with $n = 1$) brings the model "closer" to being normative. Nevertheless, $\lambda = 0$ and $n = 1$ implies non-constant discounting and therefore confronts us with a game across generations. The equilibrium outcome to that game is not normative, for the same reason that the non-cooperative Nash equilibrium to virtually any game is not normative.

4 An application to climate policy

The "climate component" of the model consists of a flow payoff and an equation of motion:

$$u(S, x_i) = -\frac{1}{2}\left(\frac{1}{n}S^2 + nx_i^2\right) \text{ and } \dot{S} = g + dS + c\left(x_i + \sum_{j \neq i} x_j\right), \quad (21)$$

where x_i and S are linear transformations of tribe i's abatement and the stock of atmospheric carbon, respectively. I explain how to obtain the linear equilibrium to this model, and then discuss its main limitations. I then describe the calibration and a transformation that causes the model to have the particular appearance shown in equation (21).

The linear equilibrium is $x = \frac{1}{n}(a + \Delta S)$, where a, Δ are functions of the model parameters. In order to obtain these functions, I treat a, Δ, as parameters and replace x_i, x_j in the equation of motion for S with the hypothesized control rule. Solving this differential equation and substituting the solution, together with the control rule for x_i, into the definition of the function

K and then integrating, results in a quadratic function, $K(S)$. Using this function, the flow payoff for the fictitious control problem is $u(S, x_i) - K(S)$; using the hypothesized control rule for agents $j \neq i$, I obtain the equation of motion for that problem, $\dot{S} = F(S, x) \equiv g + dS + c\left(x + \frac{n-1}{n}(a + \Delta S)\right)$. In a symmetric MPE, the optimal control rule obtained by solving this problem must equal $x = \frac{1}{n}(a + \Delta S)$. For the case of exponentially distributed lifetime, Δ is the solution of a cubic that satisfies an inequality to insure that $K(S)$ exists. (The solution is more complicated for the case of deterministic lifetime.) Given this value of Δ, there is a closed form expression for the constant in the control rule, a. (Appendix B.4.3.)

There is a long list of reasons why the linear-quadratic model cannot "accurately" reflect the complex problem of climate change. A one-state linear model can provide only a rough approximation of the hugely complex carbon cycle (Archer and Brovkin 2008). The quadratic utility function is obviously restrictive. The exclusion of private investment ignores a possibly important macro interaction. The stationarity of the model eliminates exogenous technical change that would result, amongst other things, in changing levels of BAU emissions. By way of compensation, the model requires only four parameters, three to describe the equation of motion and a fourth for the utility function (Appendix B.4.4). Given the amount of uncertainty about climate change, such a parsimonious model is particularly useful for the kind of fundamental question that this paper addresses.

To determine the three parameters of the equation of motion, I assume that the steady state stock of atmospheric carbon (expressed as parts per million by volume, ppm) in the absence of anthropogenic emissions equals the pre-industrial level 280; that the half-life of atmospheric carbon is 83 years; and that it would take 90 years for the year 2010 stock (380 ppm) to reach 700 ppm under constant business as usual (BAU) emissions. (The IPPC's 2007 projections of year 2100 stocks range from 535 to 983 ppm.) These assumptions imply that the steady state stock under BAU is 986 ppm. This steady state may be unrealistically high, but that exaggeration may be offset by the fact that this model ignores the possibility of catastrophic damages or abrupt changes in the equation of motion (e.g. due to a rapid melting of permafrost).

The remaining calibration parameter measures the magnitude of flow costs of stock-related damages relative to the magnitude of flow costs of abatement. There are many ways to select this parameter. Define P as the flow cost due to a doubling of the stock, relative to the pre-industrial level, and Q as the flow cost of a 50% reduction in emissions, relative to the BAU level; express both P and Q as a percent of gross world product (GWP). (The stationarity of the model implies that BAU emissions and GWP in the absence of climate damage are constant.) My fourth calibration equation sets $\frac{P}{Q} = \Omega$. Karp and Zhang (2006) estimate an abatement cost parameter that matches (with $R^2 = 0.97$ in a psuedo-regression) the cost assumptions in Nordhaus (1994). That estimate implies that Q equals about 1.1% of GWP. A low-to-moderate estimate of P is 1.33% of GWP. These estimates suggest that $\Omega = \frac{1.33}{1.12} \approx 1.2$ is consistent with (at least some) previous modeling efforts. I report results for $\Omega = 1$ (low damages) and $\Omega = 3$ (moderate damages).

I use linear transformations of the control and state variables (Appendix B.4.1) and ignore a constant in the flow payoff to write the utility function for $n = 1$ as $-\frac{1}{2}\left(S^2 + x^2\right)$; the parameter Ω that initially appears in the flow payoff "migrates" to the equation of motion: g and c are functions of Ω. I then "fragment" the agents so that the model has the appearance shown in equation (21). The three climate parameters of the transformed model are (g, d, c).

The three parameters of particular interest are: (i) λ, an inverse measure of the extent to which agents currently alive care about unborn generations in their tribe; (ii) n, a measure of the degree to which tribal fragmentation impedes agents currently alive from cooperating on current policy; and (iii) Ω, a measure of the damages from increased carbon stock relative to the costs of abatement. The remaining parameters, r (the pure rate of time preference), θ (the mortality rate), and the calibration parameters used to describe the BAU evolution of the stock, are less controversial (under the maintained hypothesis of a linear equation of motion).

4.1 Results

By definition, abatement is 0 before $t = 0$. Once the game starts, at $t = 0$, the equilibrium level of abatement is positive, and in some cases large. This discontinuity in abatement at $t = 0$ is a consequence of the calibration procedure, the absence of adjustment costs, and the fact that the game has to start at some time. The discontinuity makes the initial ($t = 0^+$) equilibrium abatement level of questionable interest. I therefore report results for the year 2100 ($t = 90$) stock. The end-of-century stock level is often used in policy discussions. The comparative statics for the initial abatement levels are similar to those for the year 2100 stocks.

The results presented here use $r = 0.02 = \theta$ (a risk-adjusted discount rate of 4%) and the model with exponentially lived paternalistic agent.[8] I use $\lambda \in \{0, 0.02, 0.1, \infty\}$ to represent a range of altruism, $n \in \{1, 5, 10\}$ to represent a range of fragmentation amongst tribes, and $\Omega \in \{1, 3\}$ to represent a range of beliefs about the cost of climate change relative to the cost of abatement.

Table 1 shows the equilibrium reduction in year 2100 stock, as a percent of the BAU level (hereafter, "abatement"). The first element corresponds to low damages, $\Omega = 1$, and the second element corresponds to moderate damages, $\Omega = 3$. For comparison, keeping the year 2100 stock to 400, 500 and 600 ppm require, respectively, decreases of 43%, 29% and 14% of the BAU levels of 700 ppm. For the parameter values here, the equilibrium year 2100 stock ranges from 470 ppm (with $n = 1$ and $\lambda = 0$) to just under 700 ppm for large values of n and λ.

$\lambda \backslash n$	1	5	10
0.0	$(19, 33)$	$(6, 12)$	$(3, 8)$
0.02	$(11, 25)$	$(2, 7)$	$(1, 4)$
0.1	$(8, 19)$	$(2, 5)$	$(1, 2)$
∞	$(7, 17)$	$(2, 4)$	$(1, 2)$

Table 1: The reduction in the year 2100 stock as a % of BAU level. The first element corresponds to $\Omega = 1$ and the second corresponds to $\Omega = 3$.

A decrease in altruism (larger λ) or contemporaneous cooperation (larger n) of course are associated with lower abatement. The facts that n and λ have different units, and that the elasticities that we care about are not constant, complicate a comparison of the equilibrium effects of these parameters. However, $\lambda = \infty$ is an upper bound, and the equilibria under $\lambda = 0.1$ and $\lambda = \infty$ are similar; thus, $\lambda = 0.1$ corresponds to a very low level of altruism. At $n = \infty$ abatement falls to 0. The abatement levels for $n = 10$ are quite small, suggesting that $n = 10$ corresponds to a high degree of fragmentation. In the discussion below, I therefore consider the intermediate values $\lambda = 0.02$ and $n = 5$ to represent "moderate" levels of altruism and

[8] Provided that λ is not "much greater" than r, and given that $T = 50$, the discount rates in the models with exponentially distributed and deterministic lifetimes are similar for many years (Corollary 2) and the resulting equilibria are also similar.

fragmentation. I emphasize the partial and the total equilibrium effects of changing the parameters to these levels, from their lower bounds $\lambda = 0$ and $n = 1$.

An increase of n from 1 to 5 causes a much larger fall in abatement, relative to an increase in λ from 0 to 0.02. Changing both parameters from their lower bound to the moderate levels causes, for $\Omega = 3$, a nearly 80% fall in equilibrium abatement; an increase in only λ causes a 24% fall in abatement and an increase in only n causes a 64% fall in abatement. By these measures, a reduction in contemporaneous cooperation has a much larger effect on the equilibrium than does a reduction in altruism. This conclusion rests on the model details and on the region of parameter space that one considers.

The equilibrium effect of a reduction in altruism is smaller than the literature cited in the Introduction, which emphasizes the social discount rate, might suggest. That literature uses an infinitely lived agent, where the discount rate (or the pure rate of time preference) measures the value that one places on both one's own and on descendants' future consumption (or utility). At least a portion of the disagreement amongst the authors cited arose from normative questions about the appropriate treatment of future generations. The parameters r and λ disentangle selfish from altruistic considerations.

Increasing Ω from 1 to 3, holding n and λ fixed, increases the percent reduction in the year 2100 stock by a factor of 2 – 3 in most cases. An increase in n from 1 to 5 leads to a substantial drop in this stock reduction, much larger in both percentage and absolute terms than the difference caused by a change of n from 5 to 10. This type of result, where cooperation breaks down rapidly at small levels of fragmentation, while higher levels of fragmentation cause only moderate additional reductions in cooperation, is familiar from industrial organization models. For example, the Cournot equilibrium price may be close to the competitive price when there are only a few firms. A larger value of n decreases the effect of changes in altruism: if the level of contemporaneous cooperation is low, the degree of altruism is rather unimportant. For example, at $n = 10$ equilibrium stock is close to BAU levels for all values of λ, so changes in λ do not have much effect.

Section 3.3 notes that this model is not normative because it is based on a game, but argues that it gets "close" to being normative for $n = 1$ and $\lambda = 0$. In view of the limitations of this model described above, I would be cautious about treating even the "quasi-normative" results corresponding to $n = 1$ and $\lambda = 0$ as policy advice. However, these results suggest a target of keeping the year 2100 stock between 470 and 570 ppm, as Ω ranges from 1 to 3. That recommendation seems quite modest, which is in keeping with the assumption that damages relative to abatement costs are low to moderate.

5 Discussion

The provision of a long-lived public good depends on the ability of contemporaneous agents to cooperate, and on their degree of altruism towards future generations. In the climate policy context, the academic economic literature has emphasized the role of the social discount rate. The popular press and other disciplines emphasize the difficulty of getting different nations to form an agreement that internalizes the effect of their individual emissions. Although limited altruism and lack of cooperation are both obstacles to effective climate policy, their relative importance is by no means obvious. This relation depends on the specifics of the problem, in particular on parameters that determine the costs and benefits of abatement and on the dynamics of the climate system.

The manner in which these specifics affect the relative importance of the two obstacles to effective policy has, I believe, not previously been studied. Embedding a differential game in an OLG setting provides a means of asking this question formally. The resulting game requires numerical analysis – but so does virtually all climate policy analysis. The numerical problems are not much greater for the game studied here than for standard optimization problems, at least for low dimensional systems. For the linear quadratic setting, the numerical problem is trivial, and the linear quadratic structure has the added advantage of requiring only a few, easily understood calibration assumptions in order to determine parameter values.

An application illustrates these methods. The strongest empirical conclusion is that the degree of contemporaneous cooperation has significantly greater effect on equilibrium policy than does the degree of intergenerational altruism.

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A Appendix A

Table 2 introduces notation used to obtain concise expressions of the discrete time discount factors. The pure rate of time preference that an agent uses to evaluate her selfish component of welfare is r, and λ is the rate she uses to evaluate the utility or welfare of future generations; $\rho = e^{-r\varepsilon}$ and $\delta = e^{-\lambda\varepsilon}$ are the corresponding discount factors. For the case of exponentially distributed lifetime, θ is the mortality = birth = hazard rate, so $\gamma = r + \theta$ is the risk-adjusted discount rate and $\alpha = e^{-\gamma\varepsilon}$ is the corresponding risk-adjusted discount factor. With a constant population normalized to 1, $b = 1 - e^{-\theta\varepsilon}$ is the mass of agents born at the end of a period of length ε.

	mortality	selfish time preference	risk adjusted discounting	altruism weight
continuous time rates	$\theta = \frac{1}{T}$	r	$\gamma = r + \theta$	λ
discrete time factors	$b = 1 - e^{-\theta\varepsilon}$	$\rho = e^{-r\varepsilon}$	$\alpha = e^{-\gamma\varepsilon}$	$\delta = e^{-\lambda\varepsilon}$

Table 2: Parameters that appear in the discount functions

A.1 Proof of Proposition 1

The term $e^{-\lambda\varepsilon}\left(2 + e^{-\theta\varepsilon}\right) = \delta(1 + b)$ appears in a formula used to prove Proposition 1. The following lemma establishes a bound on this term.

Lemma 3 *The necessary and sufficient condition for $\delta b + \delta < 1$ for all $\varepsilon \geq 0$ is $\lambda > \theta$.*

Proof. Necessity: Using the definitions of δ and b, $\delta b + \delta = e^{-\lambda\varepsilon}\left(2 - e^{-\theta\varepsilon}\right)$. The first order approximation of this expression, evaluated at $\varepsilon = 0$, is $1 + (\theta - \lambda)\varepsilon + o(\varepsilon)$. This expression is less than 1 if and only if $\lambda > \theta$. Sufficiency: Use $\frac{d\left(e^{-\lambda\varepsilon}\left(2-e^{-\theta\varepsilon}\right)\right)}{d\varepsilon} = e^{-\lambda\varepsilon}\left((\theta + \lambda)e^{-\theta\varepsilon} - 2\lambda\right)$. If $\lambda > \theta$ then $(\theta + \lambda)e^{-\theta\varepsilon} - 2\lambda < 2\lambda\left(e^{-\theta\varepsilon} - 1\right) < 0$ so $e^{-\lambda\varepsilon}\left(2 - e^{-\theta\varepsilon}\right)$ is decreasing in ε. Therefore, it is negative for all $\varepsilon \geq 0$ if and only if it is negative for $\varepsilon = 0$. ■

Proof. (Proposition 1) Substituting equation (3) into (2) gives

$$V_\tau = \sum_{s=0}^{\infty} \alpha^s u_{\tau+s} + b \sum_{s=1}^{\infty} \delta^s \left(\sum_{k=0}^{\infty} D_k u_{\tau+s+k} \right)$$

Making a change of variables, $t = s + k$ and then reversing the order of summation and simplifying yields

$$V_\tau = u_\tau + \sum_{t=1}^{\infty} \left(\alpha^t + b \sum_{s=1}^{t} \delta^s D_{t-s} \right) u_{\tau+t}. \tag{22}$$

Equating coefficients in equations (3) and (22) implies

$$D_t = \alpha^t + b \sum_{s=1}^{t} \delta^s D_{t-s} \tag{23}$$

with initial condition $D_0 = 1$. The manipulations above are valid because $\sum_{s=0}^{\infty} D(s;\varepsilon)$ is bounded, as established in part (ii) below.

An inductive proof establishes part (i). Setting $t = 0$, the trial solution in equation (4) satisfies the initial condition $D_0 = 1$. Suppose that for $t \geq 0$, the trial solution solves the recursion (23) for $\tau \leq t$. I need to show that this hypothesis implies that the trial solution solves the recursion for $t + 1$. The hypothesis implies

$$D_{t+1} = \alpha^{t+1} + b \sum_{s=1}^{t+1} \delta^s D_{t+1-s} = \alpha^{t+1} + b \sum_{s=1}^{t+1} \delta^s \frac{\alpha^{t+1-s} (\delta - \alpha) + \delta b (\delta b + \delta)^{t+1-s}}{-\alpha + \delta b + \delta}.$$

Simplifying the last expression gives

$$D_{t+1} = \frac{\alpha^{t+1} (\delta - \alpha) + \delta b (\delta b + \delta)^{t+1}}{-\alpha + \delta b + \delta},$$

as was to be shown.

(iia) The facts that $\alpha < 1$ and $(\delta b + \delta) < 1$ (from Lemma 3) imply that D_t is bounded and approaches 0 as $t \to \infty$. To show that $D_t > 0$, consider three cases, where $\delta > \alpha$, where $\alpha > \delta > \frac{\alpha}{b+1}$, and where $\delta < \frac{\alpha}{b+1}$. In the first case, the numerator and denominator of D_t are positive by inspection. In the second case, the denominator is positive. The numerator is positive iff

$$L(t) \equiv \left(\frac{\delta b + \delta}{\alpha} \right)^t > \frac{\alpha - \delta}{\delta b}. \tag{24}$$

The function $L(t)$ is increasing in t, $L(0) = 1$, and $\frac{\alpha - \delta}{\delta b} < 1$ because $\delta > \frac{\alpha}{b+1}$; therefore, inequality (24) is satisfied. Consequently, the numerator of D_t

is positive, so D_t is positive. In the third case, the denominator of D_t is negative and the numerator is negative iff

$$L(t) \equiv \left(\frac{\delta b + \delta}{\alpha}\right)^t < \frac{\alpha - \delta}{\delta b}. \tag{25}$$

Here, $L(0) = 1$ and is decreasing in t and the right side of inequality (25) is greater than 1, so the inequality is satisfied.

(iib) The fact that D is the sum of two geometrically decreasing terms, means that it's infinite sum is bounded. The sum equals $\frac{1-\delta}{(1-\delta b-\delta)(1-\alpha)}$. ■

A.2 Continuous time discount factors

In order to avoid complicating the notation, the text does not distinguish between the number of units of time (e.g. years) and the number of periods. Thus, t and τ refer to period indices in the discussion of discrete time models, and they refer to units of calendar time in the discussion of continuous time models; similarly, the text uses T to refer to the number of periods that the agent lives in the discrete time setting, and the number of units of time that she lives in the continuous time setting. For fixed ε, this notation does not present an issue, but here I want to consider the limiting case as $\varepsilon \to 0$. For this purpose, it is important to maintain the distinctions between the period index and units of time.

The text expresses the discrete time discount factors as functions of t and T. For this appendix (only) I use τ and Γ to refer exclusively to units of time and t and T to refer exclusively to number of periods. If a period lasts ε units of time, $\tau = t\varepsilon$ and $\Gamma = T\varepsilon$. Using these definitions, I set $t = \frac{\tau}{\varepsilon}$ and $T = \frac{\Gamma}{\varepsilon}$ and use the definitions in the last row of Table 1 to write the discrete time discount factors as a function of units of time (rather than number of periods) and the length of each period, ε. It is then a simple matter to take limits as $\varepsilon \to 0$.

Derivation of equation (6) Define $\mu(\epsilon) = \frac{b\delta}{\alpha-\delta} = \frac{\left(1-e^{-\theta\epsilon}\right)e^{-\lambda\epsilon}}{e^{-(r+\theta)\epsilon}-e^{-\lambda\epsilon}}$ and use L'Hospital's Rule to obtain $\lim_{\epsilon\to 0}\mu(\epsilon) = \frac{\theta}{\lambda-\theta-r} = \frac{\theta}{\lambda-\gamma}$. Using this definition of μ and the last row of Table 2, equation (1), the discount function expressed as a function of time, rather than number of periods, is

$$(1+\mu(\epsilon))\, e^{-(r+\theta)\tau} - \mu(\epsilon)\, e^{-\lambda\tau}.$$

Letting $\epsilon \rightarrow 0$ gives the continuous time discount factor for calendar time τ, equation (6). Ekeland and Lazrak (2010) obtain this formula directly (without the discrete time detour).

Proof of Proposition (2) In order to establish the claim, denote the discount rate that the agent with pure altruism applies to future generations as λ' (instead of λ, the rate under paternalistic altruism). Define the function

$$\xi(\varepsilon) = \frac{-\alpha + \delta(b+1)}{\varepsilon} = \frac{-e^{-\gamma\varepsilon} + e^{-\lambda'\varepsilon}\left(2 - e^{-\theta\varepsilon}\right)}{\varepsilon}$$

and use $\lim_{\varepsilon\rightarrow 0} \xi(\varepsilon) = r + 2\theta - \lambda'$. Also define $\phi(\varepsilon) = \frac{\ln\left(2-e^{\theta\varepsilon}\right)}{\varepsilon}$ and use $\lim_{\varepsilon\rightarrow 0} \phi(\varepsilon) = \theta$. Finally, note that

$$(b+1)^t = \left(2 - e^{-\theta\varepsilon}\right)^{\frac{\tau}{\varepsilon}} = \exp\left(\tau \frac{\ln\left(2 - e^{-\theta\varepsilon}\right)}{\varepsilon}\right) = e^{\tau\phi(\varepsilon)},$$

so $\lim_{\varepsilon\rightarrow 0} \left(2 - e^{-\theta\varepsilon}\right)^{\frac{t}{\varepsilon}} = e^{\tau\theta}$. With these definitions and the last row of Table 1, the discrete time discount factor in equation (4) can be written

$$\frac{e^{-\gamma\tau}\left(\frac{e^{-\lambda'\varepsilon} - e^{-\gamma\varepsilon}}{\varepsilon}\right) + e^{-\lambda'\varepsilon}\frac{\left(1-e^{-\theta\varepsilon}\right)}{\varepsilon} e^{-\lambda'\tau} e^{\tau\phi(\varepsilon)}}{\xi}.$$

Using the limiting expressions given above, the limit of this function as $\varepsilon \rightarrow 0$ gives the continuous time discount factor for the agent with pure altruism:

$$D(t) = \frac{e^{-\gamma\tau}\left(\gamma - \lambda'\right) + \theta e^{-(\lambda' - \theta)\tau}}{\gamma + \theta - \lambda'}. \qquad (26)$$

The right side of equations (6) and (26) are equivalent if and only if $\lambda' = \lambda + \theta$.

B Referees' Appendix

This appendix collects a number of calculations and remarks and provides details for the linear-quadratic model. It is not intended for publication.

B.1 Derivation of equation (11)

Define $\nu(\epsilon) = \frac{\delta-\rho}{\varepsilon} = \frac{\left(e^{-\lambda\epsilon}-e^{-r\epsilon}\right)}{\epsilon}$ and use $\lim_{\epsilon\to 0}\nu(\epsilon) = r-\lambda$. With this definition of ν and the last row of Table 2, the discrete time discount factor for $t<T$ can be written

$$\begin{aligned} &\tfrac{\varepsilon}{\Gamma(\delta-\rho)}\left(\delta\left(\delta^t-\rho^t\right)+(\Gamma-\tau)\,\rho^t\tfrac{(\delta-\rho)}{\varepsilon}\right) \\ &= \tfrac{e^{-\lambda\varepsilon}\left(e^{-\lambda\tau}-e^{-r\tau}\right)}{\Gamma\nu}+\tfrac{(\Gamma-\tau)}{\Gamma}e^{-r\tau}. \end{aligned}$$

Taking the limit as $\varepsilon\to 0$ and rearranging the resulting expression produces the first line of equation (11).

For $t\geq T$ the discount factor is

$$\frac{\varepsilon}{\Gamma(\delta-\rho)}e^{\lambda(\Gamma-\tau-\varepsilon)}\left(e^{-\lambda\Gamma}-e^{-r\Gamma}\right).$$

Taking the limit as $\varepsilon\to 0$ produces the second line of equation (11). The text uses T (rather than Γ) to denote the deterministic lifetime.

B.2 Equilibrium conditions for other cases

There are two cases under exponentially distributed lifetime and a singe case with a deterministic lifetime, because in the former but not in the latter, $\lim_{t\to\infty}\eta(t)$ depends on whether $\lambda<r$ or $\lambda>r$. First consider the case where lifetime is exponentially distributed lifetime and $0<\lambda<r$. Using $\lim_{t\to\infty}\eta(t)=\lambda$ from equation (10), the dynamic programming equation for the fictitious control problem is

$$\lambda J(S) = \max_x\left(U(S_t,x_t)-K(S_t)+J_S(S)\,f(S,x)\right) \tag{27}$$

and the endogenous function K equals

$$K(S_t) = (r-\lambda)\int_0^\infty e^{-\gamma t}U(S^*_{t+\tau},\chi\left(S^*_{t+\tau}\right))d\tau. \tag{28}$$

The integral in equation (28) is the present discounted value of the future flow of utility, discounted at the risk-adjusted rate γ. Here $K(S_t)$ is an annuity which if received in perpetuity and discounted at the rate $r-\lambda>0$, equals the value of this future stream of payoff. Inspection of the DPEs (19) and (27) shows that the two equations are continuous in λ and at $\lambda=r$ they collapse to the DPE for the optimal control problem with constant discount rate r and flow payoff $U(S_t,x_t)$.

For agents with deterministic lifetime T, the function K has a slightly different form than above. Using equation (4) of Karp (2007) and equations (11) and (12) above, $-K$ equals[9]

$$-K(S_t)\equiv(\lambda-r)\int_0^T\frac{(T-\tau)}{T}e^{-r\tau}U\left(S^*_{t+\tau},\chi\left(S^*_{t+\tau}\right)\right)d\tau. \qquad (29)$$

The integral on the right side of definition (29) is the present value, discounted at the selfish rate r, of the payoff that those alive at t receive from the flow $U\left(S^*_{t+\tau},\chi\left(S^*_{t+\tau}\right)\right)$ over $[t,T+t]$. Over that interval, the number of agents remaining from the time t population decreases linearly. Again, we can interpret $-K$ as an annuity, which if received in perpetuity and discounted at the rate $(\lambda-r)$, equals the present value to those currently alive of the program $U\left(S^*_{t+\tau},\chi\left(S^*_{t+\tau}\right)\right)$.

The dynamic programming equation in this case is

$$\lambda J(S)=\max_x\left(U(S_t,x_t)-K(S_t)+J_S\left(S\right)f(S,x)\right) \qquad (30)$$

with the annuity K given by equation (29).

B.3 Sufficiency

The discussion of sufficiency in Karp (2007) is misleading, and I take this opportunity to clarify it. The endogeneity of the function $K(S)$, and the resulting difficulty in determining its curvature, renders inapplicable the standard sufficiency conditions for the fictitious control problem, defined by equations (16) and (17). However, the sufficiency regarding the fictitious control problem is a red herring, because that problem is merely a device for describing

[9]Karp (2007) sets up the problem using a discount rate $r(t)$ for $t\leq T$ and $r(t)=\bar{r}$ for $t\geq T$. The results for the case of exponentially distributed lifetime use the limiting case as $T\rightarrow\infty$. In the OLG model with finitely lived agents, T is finite and $\bar{r}=\lambda$.

the equilibrium to the sequential game induced by non-constant discounting; for that purpose, we use only the necessary conditions to the fictitious problem. The maximization problem in equation (21) of Karp (2007) is merely a statement of problem for the planner in a particular period in a discrete time setting, under the assumption of Markov perfection. Equation (5) of that paper (equivalently, equation (19) above) is merely the limiting form of the discrete time condition, as the length of a period of commitment goes to 0. Therefore, provided that we are willing to restrict attention to the limiting game (as the length of a period goes to zero in the discrete time game), and provided that the value function is differentiable, a sufficient condition for the MPE is that the control rule satisfy equation (18) above, and that the value function satisfy the DPE (19).

The primitive functions of some interesting optimal control problems do not have the curvature need to satisfy familiar sufficient conditions. Sufficiency in optimal control problems is therefore sometimes a difficult issue, and the analysis sometimes proceeds without reference to sufficiency. The difficulty arises because sufficiency is a global property in optimal control problems. In contrast, sufficiency is a much simpler issue in the type of sequential game induced by non-constant discounting and the requirement of Markov perfection. In this game, each of the succession of social planers chooses a single action; given her beliefs about successors' policy function, each policy maker thus solves a static optimization problem. Because each of the policymakers treats the functions $J(S)$ and $K(S)$ as predetermined (although they are endogenous to the game), sufficiency requires (in the limit as $\varepsilon \to 0$) only that $x = \chi(S)$ maximize $(U(S_t, x_t) + J_S(S) f(S, x))$.

B.4 The linear-quadratic model with n tribes

In the interest of generality, I provide the formulae for the model that includes an interaction between the control and state variables in the utility function. The climate model does not require this interaction, so the formulae in the text are slightly simper. I first make a linear transformation of the state and control variables in order to reduce the dimension of parameter space in the linear-quadratic one-tribe model. I then convert the $n = 1$ model to the multi-tribe setting. The next subsections provide formulae for the linear equilibrium in the model for general n, under both exponentially distributed and deterministic lifetimes. I then discuss the calibration.

Each of the subsections of this part of the appendix is intended to be self-contained. In order to avoid esoteric notation, some symbols are recycled, so they have different meanings in the different subsections. Where this is the case, each subsection contains its definitions.

B.4.1 Reduction in parameter space

Here I make a linear transformation of the state and control variables that reduces the dimension of parameter space from 8 to 4. Begin with a one-tribe model in which the state variable is σ and the control variable is φ. Given the 8 parameters w, W, v, V, M, G, d, C, the flow payoff in the one-tribe setting is

$$-\left(w\varphi + W\varphi^2 + v\sigma + V\sigma^2 + M\varphi\sigma\right) \tag{31}$$

and the equation of motion is

$$\dot{\sigma} = G + d\sigma + C\varphi. \tag{32}$$

Define a new state and control variable:

$$\begin{aligned} S &= \sqrt{2V}\left(\tfrac{2vW - Mw}{4VW - M^2} + \sigma\right) \\ X &= \sqrt{2W}\left(\tfrac{2wV - Mv}{4VW - M^2} + \varphi\right) \end{aligned} \tag{33}$$

With these definitions, the flow payoff and the equation of motion are, respectively,

$$-\frac{1}{2}\left(X^2 + S^2 + mXS\right) + \text{ constant, and } \dot{S} = g + dS + cX \tag{34}$$

with $m \equiv \frac{M}{\sqrt{VW}}$ and

$$g \equiv \sqrt{2V}\left(G - D\frac{2vW - Mw}{4VW - M^2} + C\frac{Mv - 2wV}{4VW - M^2}\right) \text{ and } c \equiv \frac{C\sqrt{V}}{\sqrt{W}}. \tag{35}$$

The constant in the flow payoff does not affect behavior, so I ignore it henceforth.

B.4.2 From one tribe to n tribes

I want to define a flow payoff and an equation of motion (the technology) such that the aggregate feasible payoff does not depend directly on n. The equilibrium aggregate payoff and decision rule depends on n only insofar as n alters the equilibrium decision rules of the individual tribes. That is, n has a strategic but not an intrinsic effect on agents' payoffs.

Define the flow payoff and the constraint facing the i'th tribe as

$$-\frac{1}{2}\left(\frac{1}{n}S^2 + nx_i^2 + mx_iS\right) \qquad \dot{S} = g + dS + c\left(x_i + \sum_{j\neq i} x_j\right), \tag{36}$$

where x_i is the control variable of the i'th tribe at an arbitrary point in time. If all tribes use the same decision, x, the aggregate action is $X = \sum_i x_i = nx$. The aggregate payoff when all tribes use the same action is n times the first expression in system (36), which equals the first expression in system (34). When all tribes use the same action, the equations of motion in the two systems are obviously identical.

B.4.3 Finding the linear equilibrium

If agent *it* expects all agents to use the control rule $x = \frac{a+\Delta S}{n}$, then given the initial condition $S(t) = s$, the value of

$$S(t+\tau) = \frac{-g - ca}{d + c\Delta}\left(1 - e^{(d+c\Delta)(t+\tau)}\right) + e^{(d+c\Delta)(t+\tau)}s. \tag{37}$$

Exponentially distributed lifetime For the purpose of obtaining the linear equilibrium under exponentially distributed lifetimes for the two cases $\lambda < r$ and $\lambda > r$, and for the limiting case $\lambda = \infty$, I introduce constants η, ϵ, ν that take the values given in Table 3

	$\eta =$	$\epsilon =$	$\nu =$	$K(S) = \frac{\epsilon}{2}L(S)$
if $\lambda > r$	λ	θ	γ	$\frac{\theta}{2}L(S)$
if $\lambda < r$	γ	$\lambda - r$	λ	$\frac{\lambda - r}{2}L(S)$
if $\lambda = \infty$	n/a	$\epsilon = 0$	γ	$K(S) \equiv 0$

Table 3: values of η, ϵ, ν for different cases

I use a constant discount rate η to evaluate the integral in the function $L(S)$, defined below; I then multiply this function by $\frac{\epsilon}{2}$ to obtain the function $K(S)$; I use the constant discount rate ν to solve the fictitious control problem, where $K(S)$ is taken as given. The first step is to find the function $K(S)$

taking as given the control rule $x = \frac{a+\Delta S}{n}$ and then use that function to obtain the constants a, Δ.

Substitute the control rule $x = \frac{a+\Delta S}{n}$ into the flow payoff in equation (36) to write (negative two times) the integral of the agent's future plow payoff, discounted at a constant $\eta > 0$:

$$\begin{aligned} L(S) \equiv -2\int_0^\infty e^{-\eta t} u^* dt = \\ \int_0^\infty e^{-\eta t}\left(\tfrac{1}{n}S(t)^2 + n\left(\tfrac{a+\Delta S(t)}{n}\right)^2 + m\tfrac{a+\Delta S(t)}{n}S(t)\right)dt, \end{aligned} \tag{38}$$

where S is the value of $S(t)$ at $t = 0$. Using the value of $S(t)$ from equation (37) the integral on the right side of (38) simplifies to

$$\begin{aligned} L(S) &= \kappa_2 S^2 + \kappa_1 S + \kappa_0 \\ \kappa_2 &= -\tfrac{\left(\Delta^2+m\Delta+1\right)}{n(-\eta+2d+2c\Delta)} \\ \kappa_1 &= \kappa_{11}a + \kappa_{12} \\ \kappa_{11} &= \tfrac{\left(-\left(2\Delta^2 c-2\Delta\eta+4\Delta d-2c+2md-m\eta\right)\right)}{(-\eta+d+c\Delta)(-\eta+2d+2c\Delta)n} \\ \kappa_{12} &= \tfrac{\left(2\left(\Delta^2+m\Delta+1\right)g\right)}{(-\eta+d+c\Delta)(-\eta+2d+2c\Delta)n} \end{aligned} \tag{39}$$

The formula for κ_0 is not needed to obtain a, Δ. In the climate model, the stock decays, so $d < 0$. The control variable is abatement as a percent of BAU emissions (rather than emission), so $c < 0$. In view of these inequalities, the function $L(S)$ exists (so that the expressions for κ_i above are correct) if and only if

$$\Delta > \frac{\eta - 2d}{2c}. \tag{40}$$

Using the fact that $K(S) = \frac{\epsilon}{2}L(S)$ (by the definitions in Table 3), and assuming a quadratic value function, $J = \rho_0 + \rho_1 S + \frac{1}{2}\rho_2 S^2$, the DPE for a representative agent is

$$\begin{aligned} \nu\left(\rho_0 + \rho_1 S + \tfrac{1}{2}\rho_2 S^2\right) = \\ \max_x \left(\begin{array}{c} \left(-\frac{1}{2n} - \frac{1}{2}\epsilon\kappa_2\right)S^2 + \left(-\frac{1}{2}\epsilon\kappa_1 - \frac{1}{2}mx\right)S - \frac{1}{2}nx^2 - \frac{1}{2}\epsilon\kappa_0 + \\ (\rho_1 + \rho_2 S)\left(\left(d + c\frac{n-1}{n}\Delta\right)S + g + c\left(x + \frac{n-1}{n}a\right)\right) \end{array}\right) \end{aligned}$$

The right side of this equation is concave in x, so the first order condition provides the unique maximum. Maximization gives the control rule

$$x = c\frac{\rho_1}{n} + \frac{1}{2}\frac{-m + 2c\rho_2}{n}S \Longrightarrow a = c\rho_1 \text{ and } \Delta = \frac{-m + 2c\rho_2}{2}. \tag{41}$$

Equating coefficients S^2 and S of the maximized DPE gives

$$\nu\rho_2 = \frac{1}{4}\frac{8\rho_2 dn - 4\epsilon\kappa_2 n - 4mc\rho_2 + 8\rho_2 c\Delta n + m^2 - 8\rho_2 c\Delta + 4c^2\rho_2^2 - 4}{n}$$

$$\nu\rho_1 = \left(\frac{c^2}{n}\rho_2 + \frac{1}{8}\frac{-4mc + 8dn - 8c\Delta + 8c\Delta n}{n}\right)\rho_1 + \frac{1}{8}\frac{8gn + 8can - 8ca}{n}\rho_2 - \frac{1}{2}\epsilon\kappa_1.$$

As in the standard linear-quadratic problem, the equations for ρ_2, ρ_1 are solved recursively. Here, the coefficients of these two equations depend on a, Δ, which depend on ρ_2, ρ_1. Consequently, the equation for ρ_2 is a cubic rather than a quadratic.

The equations obtained by using the control rule in equation (41) and the definitions of coefficients κ_i in system (39) are unwieldy, so I present only the equations for $m = 0$, which holds for the climate example. With this restriction, the polynomials simplify to

$$\rho_2^3 + \frac{1}{2c^2}\frac{2(-\eta - \nu + 4d)n + \eta - 2d + \epsilon}{2n-1}\rho_2^2 + \frac{1}{2}\frac{\left(\nu\eta - 2\nu d - 2d\eta + 4d^2\right)n - 2c^2}{c^4(2n-1)}\rho_2 - \frac{1}{2}\frac{-\epsilon - \eta + 2d}{c^4(2n-1)} = 0$$

$$\rho_1 = -n\frac{2\rho_2 g - \epsilon\kappa_{12}}{-\epsilon n\kappa_{11}c + 2dn - 2\nu n - 2c\Delta + 2c^2 n\rho_2 + 2c\Delta n} \tag{42}$$

The equilibrium value of ρ_2 is a root of the cubic in the first line of (42). Any "sensible" equilibrium must have $\rho_2 < 0$; otherwise, the equilibrium payoff would be arbitrarily large for arbitrarily large stock levels. Given $\rho_2 < 0$, $\Delta = c\rho_2 > 0$ (because $c < 0$) and inequality (40) is always satisfied (because the right side of that inequality is negative). Numerically, I always find a unique negative root to the cubic. (An application of Descartes' Rule of signs might lead to an analytic proof, but applying this Rule requires signing the coefficients of the polynomial.) Once we have this value (or values) of ρ_2, we can express κ_{11}, κ_{12} and Δ as numbers, thus obtaining the value of ρ_1 and the constant a in the control rule.

Deterministic lifetime To be added

B.4.4 Calibration of the climate model

I first express the model in "natural units" and then rewrite the model so that the control and state variables are percentages that have a convenient interpretation. I then use the transformation in Appendix B.4.1 to reduce the dimension of parameter space.

Let Y_t be the stock variable at time t, ppm of carbon and x be the flow, measured in ppm per year. (One ppm by volume equals 2.13 GtC.) The equation of motion for the stock of greenhouse gases is

$$\dot{Y} = \bar{g} + \delta Y + x. \tag{43}$$

The parameters $\bar{g}$, δ, and 1 on the right side correspond to G, d, C in equation (32). To calibrate the model, I set the half life of carbon to 83 years, the steady state in the absence of anthropogenic emissions equals the pre-industrial level to 280 ppm, and assume that under BAU the stock increases from the current level, 380, to 700 in 90 years. These assumptions imply

$$\delta = -8.\,351\,170\,9 \times 10^{-3}, \quad \bar{g} = 2.\,338\,327\,8, \quad x^{\text{bau}} = 5.\,892\,686\,0. \tag{44}$$

With these parameters, the steady state stock under BAU is 986 ppm, and after 200 years of BAU the stock reaches 872 ppm.

For ease of interpretation, it is convenient to express the control variable, A, as abatement as a percent of BAU emissions, and the stock, s, as the percent increase over preindustrial levels :

$$A = \frac{x^{\text{bau}} - x}{x^{\text{bau}}} 100 \quad \text{and} \quad s = \frac{Y - 280}{280} 100$$

With these definitions and the parameter values in equation (44), the equation of motion is

$$\dot{s} = \delta s - \frac{1}{280} x^{\text{BAU}} A + \left(\frac{5}{14} x^{\text{BAU}} + 100\delta + \frac{5}{14} \bar{g} \right) = ds + CA + G \tag{45}$$

with

$$G = 2.\,104\,530\,7 \quad d = -8.\,351\,170\,9 \times 10^{-3} \quad C = -2.\,104\,530\,7 \times 10^{-2}. \tag{46}$$

In this stationary setting, denote the constant z as gross world product (GWP) exclusive of climate related damage and abatement costs. In this linear-quadratic model, the flow cost of abatement, as a percent of GWP, and the flow cost of the stock, as a percent of GWP are, respectively,

$$\frac{b}{2} \frac{A^2}{z} 100 \quad \text{and} \quad \frac{f}{2} \frac{s^2}{z} 100.$$

For calibration, suppose that the flow cost of a 50% reduction in emissions relative to BAU ($A = 50$), is Q percent of GWP; and the flow cost of doubling of ppm relative to pre-industrial level ($s = 100$), is P percent of GWP. These assumption imply

$$b = \frac{1}{125\,000} zQ, \qquad f = \frac{1}{500\,000} Pz \qquad \Longrightarrow \frac{f}{b} = \frac{P}{4Q}.$$

Define $\Omega = \frac{P}{Q}$, so that the total flow costs, as a percent of GWP, corresponding to actual abatement A and actual stock s equal

$$\left(\frac{b100}{2z}\right)\left((A)^2 + \frac{\Omega}{4}(s)^2\right).$$

Hereafter I drop the positive factor $\frac{b100}{2z}$ to write flow benefits (negative costs) as

$$-\left(A^2 + \frac{\Omega}{4} s^2\right).$$

Table 4 shows the correspondence between the variables and parameters in the general model in Section B.4.1 and in the climate model here.

	control	state								
general	φ	σ	w	W	v	V	M	G	d	C
climate	A	s	0	1	0	$\frac{\Omega}{4}$	0	G	d	C

Table 4: Correspondence between parameter values in general model from Section B.4.1 and the climate model in this section

Equation (46) gives the numerical values of G, d, C in the second row of Table 4, for this calibration. Use Table 4, the numerical values in equation (46) and the formulae in equation (35), to obtain the values of the model parameters that are used in finding the linear equilibrium:

$$g = 1.488\,127\,9\sqrt{\Omega} \quad \text{and} \quad c = -1.052\,265\,4 \times 10^{-2}\sqrt{\Omega} \tag{47}$$

In interpreting the equilibrium results, it is important to keep in mind that the model is solved in terms of the transformed state and control X,S. The percent increase in the stock relative to preindustrial level, s, and aggregate abatement as a percent of BAU emissions are related to S, X using system (33) and the correspondences in Table 3:

$$s = \sqrt{\frac{2}{\Omega}} S, \text{ and } A = X.$$

The initial condition for the problem is $s = \frac{380-280}{280} 100 = 35.714\,286$ or $S = 35.714\,286\sqrt{\frac{\Omega}{2}}$